# Knox Grammar School

## 2022

**Trial Higher School Certificate Examination**

**Name:** ____________________

**Teacher:** ____________________

# Year 12 Extension 2 Mathematics

**General Instructions**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- The official NESA Reference Sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

***Teachers:***

**Mr Bradford (Examiner)**
**Mr Vuletich**

**Section I ~ Pages 3-7**

- 10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II ~ Pages 8-15**

- 90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Write your name, your Teacher's Name and your Student Number on the front cover of each answer booklet

**Number of Students in Course: 39**

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## Section I

**10 marks**
**Attempt questions 1 – 10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1-10.

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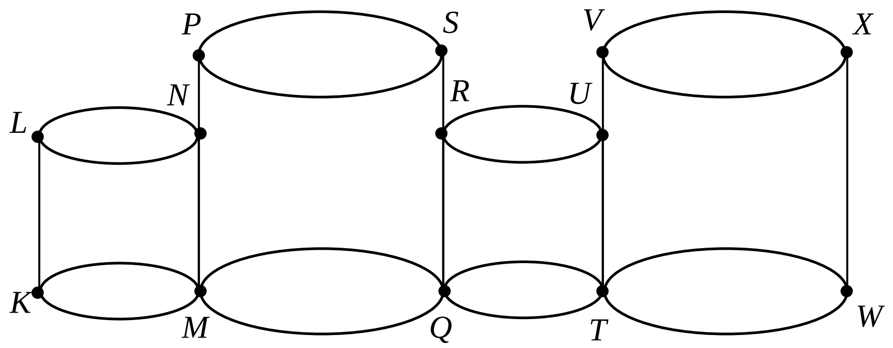
**1.** Two identical small cylinders and two identical large cylinders are placed next to each other as shown.




Which of the following is equal to $\overrightarrow{KS}$ ?

(A) $\overrightarrow{KN} + \overrightarrow{MS}$

(B) $\overrightarrow{KM} + \overrightarrow{SM}$

(C) $\overrightarrow{LN} + \overrightarrow{TX}$

(D) $\overrightarrow{SM} + \overrightarrow{MK}$

**2.** The vector equation of a line joining the points $P(1,5,3)$ and $Q(a,b,c)$ is:

$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$. Which of the following could be the coordinates of $Q$?

(A) $Q(2,2,1)$

(B) $Q(4,12,8)$

(C) $Q(4,10,8)$

(D) $Q(4,12,6)$

**3.** A particle is moving in simple harmonic motion. The displacement $x$ of the particle is given by $x = 2\sin\left(2t + \frac{\pi}{3}\right) + 4$.

Which of the following is the first time the velocity of the particle is a minimum?

(A) $\frac{\pi}{3}$

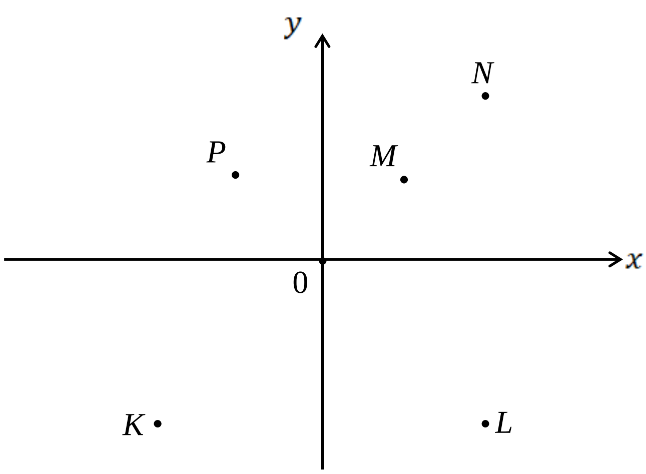
(B) $\frac{\pi}{12}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

**4.** In the Argand diagram the point $P$ represents a complex number.

When this number is multiplied by $\frac{(1+i)^4}{(1-i)^2}$ it produces a new complex number.




Which of the following points represents the new complex number?

(A) $K$

(B) $L$

(C) $M$

(D) $N$

**5.** Consider the statement: "For all quadrilaterals, if the four sides are equal in length, then the quadrilateral is a rhombus."

Which of the following is the contrapositive of the statement?

(A) For all quadrilaterals, if the four sides are not equal in length, then the quadrilateral is not a rhombus.

(B) For all quadrilaterals, if the quadrilateral is not a rhombus, then the four sides are not equal in length.

(C) There exists a quadrilateral such that the four sides are not equal in length and the quadrilateral is not a rhombus.

(D) There exists a quadrilateral such that the quadrilateral is not a rhombus and the four sides are not equal in length.

**6.** Consider the statement: "For every polynomial, if it has only three real roots, then its degree is three and its coefficients are real."

Which of the following is the negation of the statement?

(A) For every polynomial, if it has only three real roots, then its degree is three and its coefficients are not real.

(B) For every polynomial, if it has only three real roots, then its degree is not three or its coefficients are not real.

(C) There exists a polynomial that has three real roots such that its degree is not three and its coefficients are not real.

(D) There exists a polynomial that has only three real roots such that either its degree is not three or its coefficients are not real.

**7.** Given that $\omega$ is a complex root of the equation $z^3 = 1$ and $Im(\omega) > 0$ simplify:

$$\frac{\omega^2 + \frac{4}{\overline{\omega}} - \overline{\omega}}{\omega^2 + \overline{\omega}}.$$

(A) $\omega$

(B) $2\omega$

(C) $2\overline{\omega}$

(D) $\omega^2$

**8.** The roots of the cubic equation $P(x) = 0$ are $\alpha$, $\beta$ and $\gamma$.

Which of the following cubic equations has roots $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$ and $\frac{\gamma+1}{\gamma}$ ?

(A) $P\left(\frac{1}{x-1}\right) = 0$

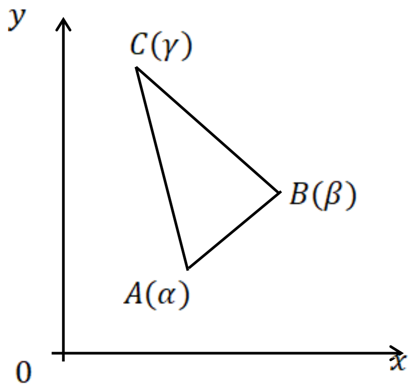
(B) $P\left(\frac{1}{x}+1\right) = 0$

(C) $P\left(\frac{1}{x}-1\right) = 0$

(D) $P\left(\frac{1}{x+1}\right) = 0$

**9.** In the diagram, $ABC$ is a triangle such that $AC = 2AB$ and $\angle BAC = \dfrac{\pi}{3}$.

The vertices $A$, $B$, and $C$ are represented by the complex numbers $\alpha$, $\beta$ and $\gamma$ respectively.




Which of the following is the correct expression of the complex number $\gamma$ ?

(A) $\gamma = \left(1+i\sqrt{3}\right)(\beta-\alpha)+\alpha$

(B) $\gamma = \left(1+i\sqrt{3}\right)(\beta+\alpha)-\alpha$

(C) $\gamma = \dfrac{1}{2}\left(1+i\sqrt{3}\right)(\beta-\alpha)+\alpha$

(D) $\gamma = \left(1+i\sqrt{3}\right)(\alpha-\beta)+\alpha$

**10.** The vertical line $x = p$, where $p$ runs through all positive integers, intersects the curve $y = e^{-x} - e^{-2x}$ at different points according to the values of $p$.

What is the limiting sum of all the $y$ coordinates for every point of intersection?

(A) $\dfrac{1}{e-1}$

(B) $\dfrac{e}{e^2-1}$

(C) $\dfrac{e+2}{e^2-1}$

(D) $\dfrac{1}{e+1}$

**End of Section I**

## Section II

**90 marks**
**Attempt questions 11 – 16**
**Allow about 2 hours 45 minutes for this section**

Answer each question in a separate writing booklet.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the complex numbers $\alpha = 1 - i$ and $\beta = -1 - i$. **2**

Find the value of $\alpha + \dfrac{1}{\alpha\beta}$, giving your answer in the form $a + ib$.

(b) A particle starts at the origin with velocity $v = 1$ and acceleration given by **3**

$a = 4v + v^2$, where $v$ is the velocity of the particle.

Find an expression for the velocity $v$ in terms of the displacement $x$.

(c) Express $\dfrac{x^2 - 5}{(x-1)\left(x^2 - x + 2\right)}$ as a sum of partial fractions with real coefficients. **3**

(d) Solve $z^2 - z + (1 - i) = 0$ giving your answers in the form $x + yi$, where $x$ and $y$ are real numbers. **4**

(e) Consider the two vectors $\underset{\sim}{a} = 8\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}$ and $\underset{\sim}{b} = \alpha\underset{\sim}{i} + \beta\underset{\sim}{k}$. **3**

Given $\underset{\sim}{b}$ is a vector perpendicular to $\underset{\sim}{a}$ and $\left|\underset{\sim}{b}\right| = 5$, find the possible values of $\alpha$ and $\beta$.

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the complex numbers $\alpha = 8e^{i\frac{5\pi}{6}}$ and $\beta = 4e^{i\frac{\pi}{6}}$. **3**

Find the value of $\dfrac{i^5\alpha^3}{2\beta^2}$, giving your answer in the form $a+bi$; $a,b \in \mathbb{R}$.

(b) Two lines, $L_1$ and $L_2$, are represented respectively by the following vector equations:

$$\underset{\sim}{r}_1 = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix} + \lambda_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \underset{\sim}{r}_2 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix}$$

The two lines intersect at the point $A(-a, a, -3a)$.

(i) Find the value of $a$ and hence write down the coordinates of $A$. **2**

(ii) Hence, find $b$ and $c$. **2**

(c) Find $\displaystyle\int \frac{\sin 2x + \cos x}{1 + \sin^2 x}\, dx$. **3**

(d) Consider the statement $P(n)$ where $n \in \mathbb{N}$. **2**

"If $n^3 + n$ is even, then $n$ is odd."

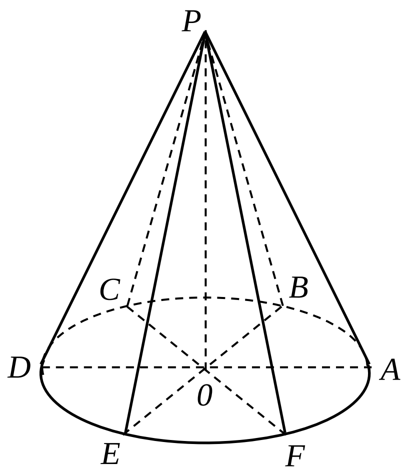
Prove that the converse of $P$ is true.

**Question 12 continues on page 10**

Question 12 (continued)

(e) The diagram shows a right-circular cone with $P$ its apex and $O$ the centre of its circular base. **3**

The points $A$, $B$, $C$, $D$, $E$ and $F$ lie on the base such that $AD$, $BE$ and $CF$ are diameters of the circular base.




Show that $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD}+\overrightarrow{PE}+\overrightarrow{PF}=6\overrightarrow{PO}$.

**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Let $I_n = \int_{-k}^{0} x^n \sqrt{x+k}\, dx$ where $n \in \mathbb{N} \cup \{0\}$ and $k > 0$. **3**

Show that $I_n = -\dfrac{2kn}{2n+3} I_{n-1}$.

(ii) Hence, evaluate $I_1$. **1**

(b) Evaluate $\displaystyle\int_0^1 \frac{3x^5}{\sqrt{x^3+1}}\, dx$. **3**

(c) (i) Show that $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$. **1**

(ii) Hence, or otherwise, find all the solutions of: **3**

$$\sin 2x + 2\sin\frac{3x}{2}\cos\frac{x}{2} + \sin x = 0 \quad \text{for} \quad -\pi \le x \le \pi$$

(d) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate $\displaystyle\int_0^{\pi/2} \frac{6}{5\sin x + 4}\, dx$. **4**

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**End of Question 13**

**Question 14** (15 marks)  Use a SEPARATE writing booklet **Marks**

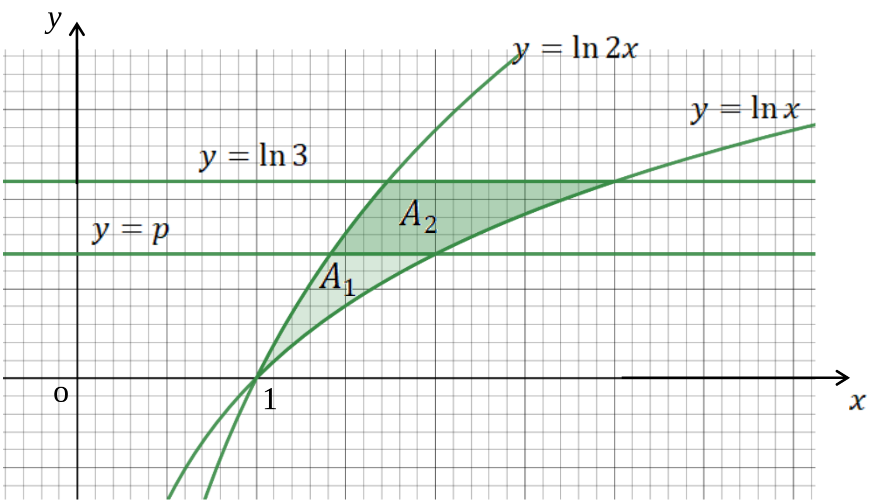
(a) Two particles $A$ and $B$ are moving in a simple harmonic motion along the $x$ axis. **3**

The displacements $x$ of the two particles $A$ and $B$ at a time $t$ are respectively given by:

$x_A = 2a\sin 4t + a\cos 4t + 9a$ and $x_B = a\sin 4t + 2a\cos 4t + a$,
where $a$ is positive real number.

Find the maximum distance between these two particles and the time at which it first occurs.

(b) The region enclosed by the curves $y = \ln x$, $y = \ln 2x$ and the line $y = \ln 3$ in the first quadrant is divided into two parts $A_1$ and $A_2$ by the line $y = p$ where $0 < p < \ln 3$.




The volumes of the solids of revolutions formed when the two areas $A_1$ and $A_2$ are rotated about the $y$ axis are respectively $V_1$ and $V_2$ and $V_2 = 3V_1$.

(i) Show that $V_1 = \dfrac{3\pi}{8}\left(e^{2p} - 1\right)$. **2**

(ii) Show that $V_2 = \dfrac{3\pi}{8}\left(9 - e^{2p}\right)$. **2**

(iii) Hence, find the exact value of $p$. **1**

**Question 14 continues on page 13**

Question 14 (continued)

(c) (i) If the equation $x^{4n-1}+i=0$ has roots $i, x_1, x_2, x_3, \cdots, x_{4n-2}$ with $i \in \mathbb{C}$, write down the roots of the equation $(x-1)^{4n-1}+i=0$. **1**

(ii) Hence, show that $(1+x_1)(1+x_2)(1+x_3)\cdots(1+x_{4n-2})=-i.$ **3**

(d) The Fundamental Theorem of Arithmetic states that every positive integer greater than one can be written as a product of prime numbers. Additionally, the prime factorisation is unique, disregarding the order in which the factors are written. **3**

Use this theorem to prove by contradiction that $\log_7 5$ is irrational.
Explain your reasoning carefully.

**End of Question 14**

**Question 15** (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) If $n$ is an odd counting number, simplify $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1}$. **1**

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(ii) Consider the proposition: **2**

"If $n$ is a positive odd integer, then $4^n + 2^n$ is divisible by 6."

Using (i) show that this proposition is true. (Do **not** use induction.)

(b) Let the complex number $\alpha = \frac{1}{2}e^{i\theta}$, where $\theta$ is real. **3**

Show that the real part of the series:

$$1 + \frac{1}{2}\alpha^3 + \frac{1}{4}\alpha^6 + \frac{1}{8}\alpha^9 + \cdots \quad \text{is} \quad \frac{256 - 16\cos 3\theta}{257 - 32\cos 3\theta}$$

(c) If $\alpha\beta\delta$ represents a three-digit number and $\alpha + \beta + \delta$ is divisible by three, show that this number is divisible by three. **2**

(d) If $a$, $b$ and $c \in \mathbb{Z}^+$, prove that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$. **3**

(e) A particle of mass $m$ kg moves under gravity in a medium in which the resistance to the motion per unit mass is $k$ times the velocity, where $k$ is a positive constant. Take $g$ as the Earth's gravitational constant.

The particle is projected vertically upwards with an initial velocity of ${}^{g}\!/_{k}$ ms$^{-1}$.

(i) Show that the velocity $v$ after time $t$ seconds is given by $kv = g\left(2e^{-kt} - 1\right)$. **2**

(ii) Show that the greatest height $H$ the particle can attain is $\frac{g}{k^2}(1 - \ln 2)$ metres. **2**

**End of Question 15**

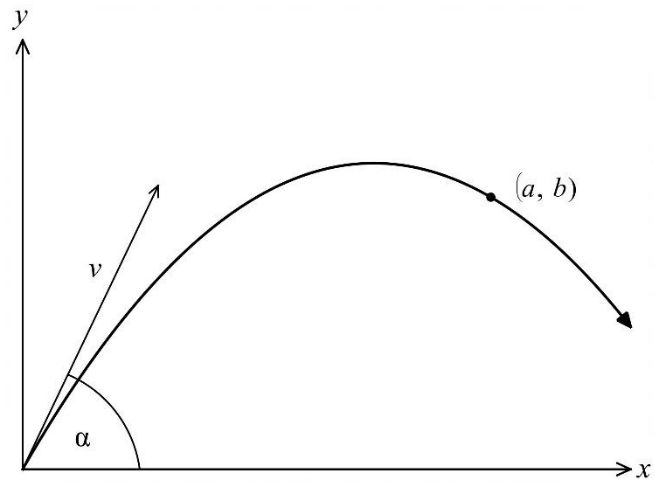
**Question 16** (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use mathematical induction to prove that $3^n \geq n^2 + 3n - 1$, for any integer $n \geq 1$. **3**

(b) Given that $\alpha$ and $\beta$ are complementary angles, prove that $\tan\alpha \tan\beta = 1$. **2**

(c) A projectile is fired from the origin with an initial velocity $v$ m/s at an angle of $\alpha$ to the positive $x$-axis. The particle is subject to acceleration due to gravity of $g$ ms$^{-2}$.

The projectile later passes through a point $(a, b)$ as shown below.




(i) Show that $\tan^2\alpha - \dfrac{2V^2 \tan\alpha}{ga} + \dfrac{2bV^2}{ga^2} + 1 = 0$. **3**

(ii) If there are two different trajectories that will allow the projectile to pass through $(a,b)$, show that $b < \dfrac{V^2}{2g} - \dfrac{ga^2}{2V^2}$. **3**

(iii) If $b = 0$, so that the range of the projectile were $a$, show that the values of $\alpha$ are complementary. **2**

(d) For arbitrary pronumerals $a, b, c, d, e, f, g$ and $h$, if $ab - gh$ is a divisor of $abef + cdgh$, show that $ab - gh$ is a divisor of $abcd + efgh$. **2**

**End of Paper**

# 2022 Year 12 Mathematics Extension 2 Task 4 Trial -Worked Solutions

**The following solutions are typical of what a competent Extension 2 student would demonstrate in an assessment setting. They are not meant to be exhaustive nor do they reflect 'best practice'. The algebra is deliberately laboured to accommodate the needs of all candidates; at least that is the goal.**

1. **C**

$\overrightarrow{KS} = \overrightarrow{KM} + \overrightarrow{MS}$

But $\overrightarrow{KM} = \overrightarrow{LN}$ and $\overrightarrow{MS} = \overrightarrow{TX}$

So $\overrightarrow{KS} = \overrightarrow{LN} + \overrightarrow{TX}$

Hence, the correct option is C.

2. **B**

Using the points $P\,(1, 5, 3)$ and $Q\,(a, b, c)$, we can find $\overrightarrow{PQ} = \begin{pmatrix} a-1 \\ b-5 \\ c-3 \end{pmatrix}$ and from the vector equation of the line $\underset{\sim}{r}$, the vector $\overrightarrow{PQ} = \lambda \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$. For $\lambda = 1$, this means

$a - 1 = 3,\; b - 5 = 7$ and $c - 3 = 5$

$a = 4 \qquad b = 12 \qquad c = 8$

So, $Q\,(4, 12, 8)$.

Hence, the correct option is B.

**Alternative method**

Check each of the four options to see which gives the same value of $\lambda$ for the components $x, y$ and $z$.

Checking option A where $Q$ is $(2, 2, 1)$.

$1 + 3\lambda = 2,\; 5 + 7\lambda = 2$ and $3 + 5\lambda = 1$

$3\lambda = 1 \qquad 7\lambda = -3 \qquad 5\lambda = -2$

$\lambda = \frac{1}{3} \qquad \lambda = -\frac{3}{7} \qquad \lambda = -\frac{2}{5}$

As there are different values of $\lambda$ then option A is invalid.

Checking option B where $Q$ is $(4, 12, 8)$.

$1 + 3\lambda = 4,\; 5 + 7\lambda = 12$ and $3 + 5\lambda = 8$

$3\lambda = 3 \qquad 7\lambda = 7 \qquad 5\lambda = 5$

$\lambda = 1 \qquad \lambda = 1 \qquad \lambda = 1$

As there is one value of $\lambda$ then option B is valid.

Note: options C and D are invalid as they will give different values for $\lambda$.

Hence, the correct option is B.

3. **A**

$x = 2\sin\left(2t + \frac{\pi}{3}\right) + 4$

$v = 4\cos\left(2t + \frac{\pi}{3}\right) = 4\cos\left(2\left(t + \frac{\pi}{6}\right)\right)$

A standard cosine curve has its first minimum to the right of the origin at $t = \pi$. As this velocity curve is dilated by a factor of $\frac{1}{2}$, that is, compressed and then translated by $\frac{\pi}{6}$ to the left, then its first minimum to the right of the origin for $t$ positive is at $\pi \times \frac{1}{2} - \frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$.

Hence, the correct option is A.

**Alternative method**

The minimum velocity occurs when $\frac{\mathrm{dv}}{\mathrm{dt}} = 0$ and $\frac{\mathrm{d^2v}}{\mathrm{dt^2}} > 0$.

$\frac{\mathrm{dv}}{\mathrm{dt}} = -8\sin\left(2t + \frac{\pi}{3}\right)$ and $\frac{\mathrm{d^2v}}{\mathrm{dt^2}} = -16\cos\left(2t + \frac{\pi}{3}\right)$

Let $\frac{dv}{dt} = 0$, we get

$-8\sin\left(2t + \frac{\pi}{3}\right) = 0$ that is $\sin\left(2t + \frac{\pi}{3}\right) = 0$

$2t + \frac{\pi}{3} = 0,\; \pi,\; 2\pi, \ldots$

$2t = -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \ldots$

$t = -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \ldots$

As $t > 0$ we test $t = \frac{\pi}{3}$, $\frac{\mathrm{d^2v}}{\mathrm{dt^2}} = -16\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = 16$. As $\frac{\mathrm{d^2v}}{\mathrm{dt^2}} > 0$ when $t = \frac{\pi}{3}$ then the first minimum turning point occurs at $t = \frac{\pi}{3}$.

Hence, the correct option is A.

4. **D**

$1 + i = \sqrt{2}\,cis\,\frac{\pi}{4}$ so $(1+i)^4 = \left(\sqrt{2}\right)^4 cis\,\frac{4\pi}{4} = 4\,cis\,\pi$

$1 - i = \sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$ so $(1-i)^2 = \left(\sqrt{2}\right)^2 cis\left(2 \times -\frac{\pi}{4}\right)$

$= 2\,cis\left(-\frac{\pi}{2}\right)$

Hence, $\frac{(1+i)^4}{(1-i)^2} = 4\,cis\,\pi \div 2\,cis\left(-\frac{\pi}{2}\right) = 2\,cis\left(\frac{3\pi}{2}\right)$

So, the new complex number will be twice as far from the origin as *P*. Also, the new complex number will have an argument which is rotated $\frac{3\pi}{2}$ $radians$ from *A*.
Hence, the correct option is D.

5. **B**

For a statement "For all, if P then Q " its contrapositive statement is "For all, if not Q then not P ".

So, as the statement is "For all quadrilaterals, if the four sides are equal in length, then the quadrilateral is a rhombus", the contrapositive statement would be: "For all quadrilaterals, if the quadrilateral is not a rhombus, then the four sides are not equal in length."

Hence, the correct option is B.

6. **D**

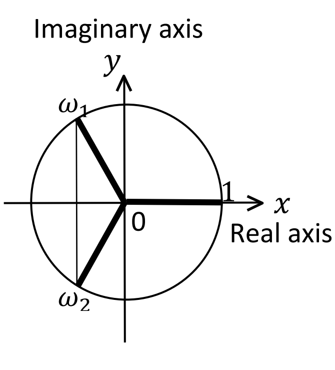
For a statement " For every, if P then Q and then R" its negation statement is " There exists a P such that either not Q *or* then not R ". Symbolically, we have:

$$\sim\left(\forall\ polynomials, P \rightarrow (Q \cap R)\right)$$
$$\equiv \exists\ polynomial, \sim\left(P \rightarrow (Q \cap R)\right)$$
$$\equiv \exists\ polynomial, P \cap \sim(Q \cap R)$$
$$\equiv \exists\ polynomial, P \cap (\sim Q \cup \sim R)\ de\ Morgan's\ Laws$$

So, as the statement is "For every polynomial, if it has only three real roots, then its degree is three and its coefficients are real", the negation statement would be "There exists a polynomial that has only three real roots such that either its degree is not three *or* its coefficients are not real."

Hence, the correct option is D.

**7. C**




The diagram above shows the three roots of the equation $z^3 = 1$ two of these roots are $\omega_1 = cis\left(\frac{2\pi}{3}\right)$ and $\omega_2 = cis\left(-\frac{2\pi}{3}\right)$ $or$ $cis\left(\frac{4\pi}{3}\right)$.

If $\omega = \omega_1 = cis\left(\frac{2\pi}{3}\right)$, then $\omega^2 = cis\left(\frac{4\pi}{3}\right) = \omega_2$.

and if $\overline{\omega} = cis\left(-\frac{2\pi}{3}\right) = cis\left(\frac{4\pi}{3}\right)$ $then$ $\overline{\omega} = \omega^2$.

Now, Let $K = \dfrac{\omega^2 + \frac{4}{\overline{\omega}} - \overline{\omega}}{\omega^2 + \overline{\omega}} = \dfrac{\omega^2 + \frac{4}{\omega^2} - \omega^2}{\omega^2 + \omega^2}$.

By multiplying both numerator and denominator $\omega^2$, we get:

$k = \dfrac{\frac{4}{\omega^2}}{2\,\omega^2} = \dfrac{4}{2\omega^4} = \dfrac{2}{\omega^4}$. As $\omega$ is a root of $z^3 = 1$ this means $\omega^{\ 3} = 1$ that is $\omega^{\ 4} = \omega$.

This indicates that $k = \dfrac{2}{\omega} = \dfrac{2\overline{\omega}}{\omega\,\overline{\omega}} = \dfrac{2\overline{\omega}}{|\,\omega|^2}$.

But $|\,\omega| = 1$ as the roots lie on a circle of radius 1, so $k = 2\overline{\omega}$.

Hence, the correct option is C.

*OR*

A simpler approach recognises $\omega^3 = 1$ $and$ $\omega\overline{\omega}$=1.

Therefore, $\dfrac{\omega^2 + \frac{4}{\overline{\omega}} - \overline{\omega}}{\omega^2 + \overline{\omega}} = \dfrac{\omega^2+4\omega-\frac{1}{\omega}}{\omega^2+\frac{1}{\omega}} = \dfrac{\omega^2+4\omega-\omega^2}{\omega^2+\omega^2}$

$= \dfrac{4\omega}{2\omega^2} = \dfrac{2}{\omega} = 2\ \overline{\omega}$. So option C.

8. **A**

We know that $P(x) = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

To find required the cubic equation, we let $y = \dfrac{x+1}{x}$ and we make $x$ the subject.

So, $xy = x + 1$ that is $xy - x = 1$

$$x(y-1) = 1$$
$$x = \frac{1}{y-1}$$

Hence, $P\left(\dfrac{1}{y-1}\right) = 0$ has roots $\dfrac{\alpha+1}{\alpha}$, $\dfrac{\beta+1}{\beta}$, $\dfrac{\gamma+1}{\gamma}$ but as the variable is immaterial, we can state that the equation is $P\left(\dfrac{1}{x-1}\right) = 0.$

Hence, the correct option is A.

**Alternative method**

As the roots $\dfrac{\alpha+1}{\alpha}$, $\dfrac{\beta+1}{\beta}$, $\dfrac{\gamma+1}{\gamma}$ should satisfy the required cubic equation we can check the equations in the options to see which one of them satisfies the roots provided.

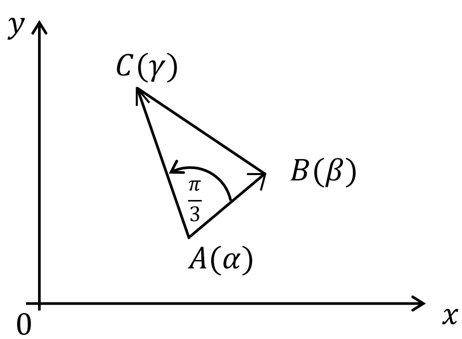
Checking option A by substituting one of the roots for example $\frac{\alpha+1}{\alpha}$, we get:

$$P\left(\frac{1}{x-1}\right) = P\left(\frac{1}{\frac{\alpha+1}{\alpha}-1}\right)$$
$$= P\left(\frac{1}{\frac{\alpha+1-\alpha}{\alpha}}\right)$$
$$= P\left(\frac{1}{\frac{1}{\alpha}}\right)$$
$= P(\alpha) = 0$ as α is a root of $P(x) = 0$

So, the root $\frac{\alpha+1}{\alpha}$ satisfies $P\left(\frac{1}{x-1}\right) = 0$.

Hence, the correct option is A.

**9. A**




By rotating vector $\overrightarrow{AB}$ by $\frac{\pi}{3}$ and multiplying its magnitude by 2 we obtain vector $\overrightarrow{AC}$. This means

$\overrightarrow{AC} = \overrightarrow{AB} \times 2 \times \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\overrightarrow{AC} = (\beta - \alpha) \times 2\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$

$\overrightarrow{AC} = (\beta - \alpha)\left(1 + i\sqrt{3}\right)$

But $\overrightarrow{AC} = \gamma - \alpha$ this indicates that

$\gamma - \alpha = (\beta - \alpha)\left(1 + i\sqrt{3}\right)$

So, $\gamma = \alpha + (\beta - \alpha)\left(1 + i\sqrt{3}\right)$

Hence, the correct option is A.

**10. B**

As $p$ is any positive integer this means $p = 1, 2, 3, \ldots$

we substitute these values of $p$ into the equation of the curve $y = e^{-x} - e^{-2x}$ to get all the $y$ coordinates of all the points of intersection. This means

when $p = 1, y = e^{-1} - e^{-2}$

when $p = 2, y = e^{-2} - e^{-4}$

when $p = 3, y = e^{-3} - e^{-6}$

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The sum of all the $y$ coordinates is

$S_\infty = e^{-1} - e^{-2} + e^{-2} - e^{-4} + e^{-3} - e^{-6} + \ldots$

This sum can be arranged into two infinite sums

$S_1 = e^{-1} + e^{-2} + e^{-3} + \cdots$ and

$S_2 = -e^{-2} - e^{-4} - e^{-6} - \cdots$

$S_1$ is an infinite geometric series with the first term $e^{-1}$and common ratio $r = e^{-2} \div e^{-1} = e^{-1}$

so, $S_1 = \frac{e^{-1}}{1-e^{-1}} = \frac{e}{e} \times \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1}$

Now, $S_2 = -e^{-2} - e^{-4} - e^{-6} - \cdots$

Also, $S_2$ is an infinite geometric series with the first term $-e^{-2}$and $r = -e^{-4} \div -e^{-2} = e^{-2}$.

so, $S_2 = \frac{-e^{-2}}{1-e^{-2}} = \frac{e^2}{e^2} \times \frac{-e^{-2}}{1-e^{-2}} = \frac{-1}{e^2-1} = \frac{-1}{(e+1)(e-1)}$

Hence, the sum all the $y$ coordinates of all the points of intersection is

$$S = \frac{1}{e-1} - \frac{1}{(e+1)(e-1)}$$
$$= \frac{e+1-1}{(e+1)(e-1)}$$
$$= \frac{e}{(e+1)(e-1)} = \frac{e}{e^2-1}$$

Hence, the correct option is B.

## Question 11

a) $\alpha = 1 - i$ and $\beta = -1 - i$

So, $\alpha\beta = (1 - i)(-1 - i)$

$= -1 - i + i - 1 = -2.$

This means $\alpha + \frac{1}{\alpha\beta} = 1 - i - \frac{1}{2} = \frac{1}{2} - i.$

b) The acceleration of the particle is $a = v\,\frac{dv}{dx}$.

This means $v\,\frac{dv}{dx} = 4v + v^2$ that is

$\frac{dv}{dx} = 4 + v$

$\int \frac{dv}{4+v} = \int dx$

$\ln|4 + v| = x + c$

When $t = 0, x = 0$ and $v = 1$

So, $\ln 5 = 0 + c$ that is $c = \ln 5$

So, $\ln|4 + v| = x + \ln 5$

$\ln|4 + v| - \ln 5 = x$

$\mathrm{Ln}\left|\frac{4+v}{5}\right| = x \;\; and\ v \neq 0, otherwise\ v = 0\ \&\ a = 0.$

This would mean the particle comes to rest permanently. In particular, since the velocity is initially positive, it would be impossible for $v < 0$. Consequently, the acceleration is always positive for all $t$.

This means$\left|\frac{4+v}{5}\right| = \frac{4+v}{5}$.

So, $e^x = \frac{4+v}{5}$ that is $5e^x = 4 + v$. Hence, $v = 5e^x - 4$.

c) As $x^2 - x + 2$ has no real factors,

let $\frac{x^2 - 5}{(x-1)(x^2 - x + 2)} = \frac{a}{x-1} + \frac{bx+c}{x^2 - x + 2}$.

Multiplying both sides by $(x-1)(x^2 - x + 2)$, we get

$x^2 - 5 = a(x^2 - x + 2) + (bx + c)(x - 1)$

For $x = 1$, we get $-4 = 2a$ that is $a = -2$.

Now, $x^2 - 5 = -2(x^2 - x + 2) + (bx + c)(x - 1)$

$x^2 - 5 = -2x^2 + 2x - 4 + (bx + c)(x - 1)$

$3x^2 - 2x - 1 = bx^2 + cx - bx - c$

By equating coefficients of $x^2$, we get $b = 3$.

and by equating constants, we get $c = 1$.

Hence, $\frac{x^2 - 5}{(x-1)(x^2 - x + 2)} = \frac{-2}{x-1} + \frac{3x+1}{x^2 - x + 2}$.

d) $z^2 - z + (1 - i) = 0$

Using the quadratic formula,

$z = \frac{1 \pm \sqrt{1 - 4(1-i)}}{2}$

$z = \frac{1 \pm \sqrt{1 - 4 + 4i}}{2}$

$z = \frac{1 \pm \sqrt{-3 + 4i}}{2}$ (1)

Now, let $(a + ib)^2 = -3 + 4i$

$a^2 - b^2 + 2ab\,i = -3 + 4i$

Equating reals, we get

$a^2 - b^2 = -3$ (1)

Equating imaginaries, we get

$ab = 2$ (2)

Equating moduli, we get

$a^2 + b^2 = 5$ (3)

(1) + (2) gives

$2a^2 = 2$ that is $a^2 = 1$

So, $a = \pm 1$

Now, when $a = 1, b = 2$ and

when $a = -1, b = -2$

So, $\sqrt{-3 + 4i} = \pm(1 + 2i)$ (2)

By substituting (2) in (1), we get

$z = \frac{1 \pm (1 + 2i)}{2}$

$z = \frac{2 + 2i}{2}$ or $z = \frac{-2i}{2} = -i$

$z = 1 + i$ or $-i$

e) $\underset{\sim}{a} = 8\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}$ and $\underset{\sim}{b} = \alpha\,\underset{\sim}{i} + \beta\underset{\sim}{k}$.

As $\underset{\sim}{a}$ is perpendicular $\underset{\sim}{b}$ then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 8\alpha + 0 - 6\beta = 0$. This means

$6\beta = 8\alpha$ that is $\beta = \frac{4\alpha}{3}$.

Also, as $|\underset{\sim}{b}| = 5$ *and this means*

$\sqrt{\alpha^2 + \beta^2} = 5$

$\alpha^2 + \beta^2 = 25$ but $\beta = \frac{4\alpha}{3}$. This indicates

$\alpha^2 + \frac{16\alpha^2}{9} = 25$

Multiplying both sides by 9, we get

$9\alpha^2 + 16\alpha^2 = 225$

$25\alpha^2 = 225$, that is $\alpha^2 = 9$ and so, $\alpha = \pm 3$.

Now, when $\alpha = 3$, $\beta = \frac{4}{3} \times 3 = 4$ and

when $\alpha = -3$, $\beta = \frac{4}{3} \times -3 = -4$

Hence, the possible values are

$\alpha = 3, \beta = 4$ or $\alpha = -3, \beta = -4$.

**Question 12**

a) $\alpha = 8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ and $i^5 = i^4 \times i = i$

*So,* $i^5\alpha^3 = i \times 8^3\left(\cos\left(3 \times \frac{5\pi}{6}\right) + i\sin\left(3 \times \frac{5\pi}{6}\right)\right)$

$i^5\alpha^3 = i \times 8^3\left(\cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right)$

$i^5\alpha^3 = i \times 8^3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$i^5\alpha^3 = i \times 8^3 \times i = -512$

Also, $\beta = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$2\beta^2 = 2 \times 4^2\left(\cos\left(2 \times \frac{\pi}{6}\right) + i\sin\left(2 \times \frac{\pi}{6}\right)\right)$

$2\beta^2 = 2 \times 4^2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

Now, $\frac{i^5\alpha^3}{2\beta^2} = \frac{-512}{32\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

$= \frac{-16}{\text{cis}\left(\frac{\pi}{3}\right)} = -16\,\text{cis}\left(-\frac{\pi}{3}\right)$

$= -16\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$= -16\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

Hence, $\frac{i^5\alpha^3}{2\beta^2} = -8 + 8\sqrt{3}\,i$

b) i) The line $\underset{\sim}{r_2} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + \lambda_2\begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix}$ passes through the point of intersection $A(-a, a, -3a)$ this means

$-1 + 2\lambda_2 = -a$ (1)

and $2 - \lambda_2 = a$

that is $4 - 2\lambda_2 = 2a$ (2).

By adding (1) and (2), we get $a = 3$, and substituting this value of $a$ in A $(-a, a, -3a)$, we get $A(-3, 3, -9)$.

Note: To check if the coordinates of $A$ are valid, we can obtain $\lambda_2$ by substituting $a = 3$ in (1), we get $-1 + 2\lambda_2 = -3$ that is $\lambda_2 = -1$ and by substituting $\lambda_2 = -1$ in $\underset{\sim}{r_2}$ we get the point

$$\begin{pmatrix}-1\\2\\-2\end{pmatrix}-1\begin{pmatrix}2\\-1\\7\end{pmatrix}=\begin{pmatrix}-3\\3\\-9\end{pmatrix}$$ which is $A$.

ii) Now, by substituting $a=3$ in the line $\underset{\sim}{r_1}$, we get

$\underset{\sim}{r_1}=\begin{pmatrix}6\\0\\-3\end{pmatrix}+\lambda_1\begin{pmatrix}3\\b\\c\end{pmatrix}$. Now, as the line $r_1$ passes through $A\,(-3, 3, -9)$ this means $6+3\lambda_1=-3$ that is $\lambda_1=-3$. Also, $0-3b=3$ that is $b=-1$ and $-3-3c=-9$ that is $c=2$. Hence, the vector equation of line $L_1$ is $r_1=\begin{pmatrix}6\\0\\-3\end{pmatrix}+\lambda_1\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

c) $\int\frac{\sin 2x+\cos x}{1+\sin^2 x}\,dx=\int\frac{\sin 2x}{1+\sin^2 x}\,dx+\int\frac{\cos x}{1+\sin^2 x}\,dx$

Let us consider first $\int\frac{\sin 2x}{1+\sin^2 x}\,dx$

As $\frac{\mathrm{d}}{\mathrm{dx}}(1+\sin^2 x)=2\sin x\cos x=\sin 2x$

Then $\int\frac{\sin 2x}{1+\sin^2 x}\,dx=\ln|1+\sin^2 x\,|+d$

As $1+\sin^2 x>0$ for all x, then

$|1+\sin^2 x\,|=1+\sin^2 x$

$\int\frac{\sin 2x}{1+\sin^2 x}\,dx=\ln(\,1+\sin^2 x\,)+d.$

Also, for $\int\frac{\cos x}{1+\sin^2 x}\,dx$ let $u=\sin x$

so $du=\cos x\,dx$

So $\int\frac{\cos x}{1+\sin^2 x}\,dx=\int\frac{du}{1+u^2}=\tan^{-1}u$ +c

$=\tan^{-1}(\sin x)+k$

Hence, $\int\frac{\sin 2x+\cos x}{1+\sin^2 x}\,dx$

$$=\int\frac{\sin 2x}{1+\sin^2 x}\,dx+\int\frac{\cos x}{1+\sin^2 x}\,dx$$

$=\ln(1+\sin^2 x)+\tan^{-1}(\sin x)+c$

d) To prove the converse of P is true this means we must prove if $n$ is odd, then $n^3+n$ is even.

let, $n=2p+1, where\ p$ is an integer.

So, $n^3+n=\ (2p+1)^3+2p+1$

$=8p^3+12p^2+6p+1+2p+1$

$=8p^3+12p^2+8p+2$

$=2\,(4p^3+6p^2+4p+1)$

Let $4p^3+6p^2+4p+1=M, where\ p$ is an integer.

This indicates thar $n^3+n=\ 2M$ which is even.

Hence, the converse of P is true.

e) From the diagram

$\overrightarrow{PA}=\overrightarrow{PO}+\overrightarrow{OA}$ and $\overrightarrow{PD}=\overrightarrow{PO}+\overrightarrow{OD}$ this means

$\overrightarrow{PA}+\overrightarrow{PD}=\overrightarrow{PO}+\overrightarrow{OA}+\overrightarrow{PO}+\overrightarrow{OD}$

Now, as $\overrightarrow{OA}\ and\ \overrightarrow{OD}$ are equal radii, but in opposite directions this means $\overrightarrow{OD}=-\overrightarrow{OA}$ and this indicates that $\overrightarrow{PA}+\overrightarrow{PD}=\overrightarrow{PO}+\overrightarrow{OA}+\overrightarrow{PO}-\overrightarrow{OA}=2\overrightarrow{PO}$ (1)

Similarly, $\overrightarrow{PB}=\overrightarrow{PO}+\overrightarrow{OB}$ and $\overrightarrow{PE}=\overrightarrow{PO}+\overrightarrow{OE}$

this means $\overrightarrow{PB}+\overrightarrow{PE}=\overrightarrow{PO}+\overrightarrow{OB}+\overrightarrow{PO}+\overrightarrow{OE}$

and as $\overrightarrow{OE}=-\overrightarrow{OB}$ (equal radii, but in opposite directions) then $\overrightarrow{PB}+\overrightarrow{PE}=2\overrightarrow{PO}$ (2)

Also, $\overrightarrow{PC}=\overrightarrow{PO}+\overrightarrow{OC}$ and $\overrightarrow{PF}=\overrightarrow{PO}+\overrightarrow{OF}$. This means $\overrightarrow{PC}+\overrightarrow{PF}=\overrightarrow{PO}+\overrightarrow{OC}+\overrightarrow{PO}+\overrightarrow{OF}$ and as $\overrightarrow{OF}=-\overrightarrow{OC}$ (equal radii, but in opposite directions) then

$\overrightarrow{PC}+\overrightarrow{PF}=2\overrightarrow{PO}$ (3)

By adding (1), (2)and (3), we get

$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD}+\overrightarrow{PE}+\overrightarrow{PF}=6\,\overrightarrow{PO}$

## Question 13

a) i) $I_\mathrm{n}=\int_{-\mathrm{k}}^{0}x^n\sqrt{x+k}\ \mathrm{dx}$

Using integration by parts.

Let $u=x^n$ and $dv=\sqrt{x+k}\,dx$

$du=nx^{n-1}\,dx$ $\quad v=\int(x+k)^{\frac{1}{2}}\,dx$

$v=\frac{2}{3}(x+k)^{\frac{3}{2}}$

$I_\mathrm{n}=\left[x^n\times\frac{2}{3}(x+k)^{\frac{3}{2}}\right]_{-k}^{0}-\int_{-\mathrm{k}}^{0}\frac{2}{3}(x+k)^{\frac{3}{2}}\times nx^{n-1}\ \mathrm{dx}$

$I_\mathrm{n}=0-\int_{-\mathrm{k}}^{0}\frac{2}{3}(x+k)(x+k)^{\frac{1}{2}}\times nx^{n-1}\ \mathrm{dx}$

$I_\mathrm{n}=-\frac{2n}{3}\int_{-\mathrm{k}}^{0}(x+k)(x+k)^{\frac{1}{2}}\times x^{n-1}\ \mathrm{dx}$

$I_\mathrm{n}=-\frac{2n}{3}\left[\int_{-\mathrm{k}}^{0}x(x+k)^{\frac{1}{2}}\times x^{n-1}\ \mathrm{dx}+\int_{-\mathrm{k}}^{0}k(x+k)^{\frac{1}{2}}\times x^{n-1}\ \mathrm{dx}\right]$

$I_\mathrm{n}=-\frac{2n}{3}\left[\int_{-\mathrm{k}}^{0}(x+k)^{\frac{1}{2}}\times x^{n}\,\mathrm{dx}+k\int_{-\mathrm{k}}^{0}(x+k)^{\frac{1}{2}}\times x^{n-1}\ \mathrm{dx}\right]$

$I_\mathrm{n}=-\frac{2n}{3}(I_\mathrm{n}+k\,I_{\mathrm{n}-1})$

$3I_\mathrm{n}=-2n\,I_\mathrm{n}-2kn\,I_{\mathrm{n}-1}$

$3I_\mathrm{n}+2n\,I_\mathrm{n}=-2kn\,I_{\mathrm{n}-1}$

$(3\,+\,2n)\,I_\mathrm{n}=-2kn\,I_{\mathrm{n}-1}$

Hence, $I_n=\frac{-2kn}{2n+3}\,I_{n-1.}$

a) ii) $I_1=-\frac{2k}{5}I_0$ and $I_0=\int_{-k}^{0}\sqrt{x+k}\,dx=\frac{2}{3}(x+k)^{3/2}\Big|_{-k}^{0}$

$$=\frac{2}{3}k^{3/2}$$

$$\therefore\ I_1=-\frac{2k}{5}\times\frac{2}{3}k^{3/2}=-\frac{4}{15}k^{5/2}$$

b) Let $u = x^3 + 1$ so $u - 1 = x^3$

$du = 3x^2\,dx$

When $x = 1, u = 2$ and when $x = 0, u = 1$.

Now, $\int_0^1 \frac{3x^5}{\sqrt{x^3+1}}\,dx$

$= \int_0^1 \frac{x^3}{\sqrt{x^3+1}} \times 3x^2\,dx$

$= \int_1^2 \frac{u-1}{\sqrt{u}}\,du$

$= \int_1^2 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)\,du$

$= \int_1^2 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\,du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2\,u^{\frac{1}{2}}\right]_1^2$

$= \left(\frac{2}{3} \times 2\sqrt{2} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right)$

$= \left(\frac{4\sqrt{2} - 6\sqrt{2}}{3}\right) - \left(-\frac{4}{3}\right) = \frac{4}{3} - \frac{2\sqrt{2}}{3} = \frac{4 - 2\sqrt{2}}{3}.$

c) i) $\sin(\alpha + \beta) + \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= 2\sin\alpha\cos\beta$

ii) $\sin 2x + 2\sin\frac{3x}{2}\cos\frac{x}{2} + \sin x = 0$

Note: from part (i) we can write

$2\sin\frac{3x}{2}\cos\frac{x}{2} = \sin\left(\frac{3x}{2} + \frac{x}{2}\right) + \sin\left(\frac{3x}{2} - \frac{x}{2}\right)$

$= \sin 2x + \sin x$

So, the equation will be

$\sin 2x + \sin 2x + \sin x + \sin x = 0$

$2\sin 2x + 2\sin x = 0$

$4\sin x\cos x + 2\sin x = 0$

$2\sin x\,(2\cos x + 1) = 0$

$\sin x = 0$ *or* $\cos x = -\frac{1}{2}$

So, the solutions are

$x = -\pi,\ 0,\ \pi$ *or* $x = -\frac{2\pi}{3},\ \frac{2\pi}{3}$ in the domain $-\pi \le x \le \pi$.

d) $\int_0^{\frac{\pi}{2}} \frac{6}{5\sin x + 4}\,dx$ Let $t = \tan\frac{x}{2}$

so $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ that is $dt = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)dx$

$dt = \frac{1}{2}(1 + t^2)dx$

$dx = \frac{2}{1+t^2}\,dt$

When $x = \frac{\pi}{2}$, $t = 1$ and when $x = 0$, $t = 0$

Also, $5\sin x + 4 = 5 \times \frac{2t}{1+t^2} + 4$

$= \frac{10t}{1+t^2} + 4$

$= \frac{10t + 4 + 4t^2}{1 + t^2} = \frac{2\,(2t^2 + 5t + 2)}{1 + t^2}$

This means $\frac{6}{5\sin x + 4} = \frac{6}{\frac{2\left(2t^2+5t+2\right)}{1+t^2}}$

$= \frac{3\,(1+t^2)}{2t^2 + 5t + 2}$

So $\int_0^{\frac{\pi}{2}} \frac{6}{5\sin x + 4}\,dx = \int_0^1 \frac{3\,(1+t^2)}{2t^2+5t+2} \times \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{6}{2t^2+5t+2}\,dt$

$= \int_0^1 \frac{6}{(2t+1)(t+2)}\,dt$

Now, let $\frac{6}{(2t+1)(t+2)} = \frac{A}{(2t+1)} + \frac{B}{(t+2)}$

Multiplying both sides by $(2t+1)(t+2)$, we get

$6 = A(t+2) + B(2t+1)$

When $t = -2$, *we get* $6 = -3B$ *so* $B = -2$.

When $t = -\frac{1}{2}$, *we get* $6 = \frac{3}{2}A$ *so* $A = 4$.

Hence, $\int_0^1 \frac{6}{(2t+1)(t+2)}\,dt$

$= \int_0^1 \left(\frac{4}{2t+1} - \frac{2}{t+2}\right)dt$

$= [\,2\ln|2t+1| - 2\ln|t+2|\,]_0^1$

$= (2\ln 3 - 2\ln 3) - (2\ln 1 - 2\ln 2)$

$= 2\ln 2$

**Question 14**

a) From the equations of the displacements of the two particles, we can see the particle *A* is always to the right of particle *B.* This means to find the distance between the two particles we subtract $x$ of *B* from $x$ of *A*.

The expression for the distance between them is

$D = 2a\sin 4t + a\cos 4t + 9a - (a\sin 4t + 2a\cos 4t + a)$

$D = a\sin 4t - a\cos 4t + 8a$

By using the auxiliary method, we can express

$a\sin 4t - a\cos 4t$ as $r\sin(4t - \alpha)$.

$r = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$ and

$\tan\alpha = \frac{a}{a} = 1$ and as $\alpha$ is an acute angle then $\alpha = \frac{\pi}{4}$.

So, $D = a\sqrt{2}\sin\left(4t - \frac{\pi}{4}\right) + 8a$

$D = a\sqrt{2}\sin 4\left(t - \frac{\pi}{16}\right) + 8a$

The sine curve in equation *D* has a period of $T = \frac{2\pi}{4} = \frac{\pi}{2}$ and it is translated horizontally by $t = \frac{\pi}{16}$ to the right.

This means the first maximum of D occurs at a horizontal distance or time $t = \frac{\pi}{16} + \frac{1}{4} \times T = \frac{\pi}{16} + \frac{1}{4} \times \frac{\pi}{2} = \frac{3\pi}{16}$ and its *y* coordinates or the *D* distance between the particles will be equal to the amplitude $a\sqrt{2}$ plus the vertical translation $8a$, that is $a\sqrt{2} + 8a = (\sqrt{2} + 8)a$.

**Alternative method**

To find the maximum distance $D$ or the maximum turning points, we differentiate $D = a\sin 4t - a\cos 4t + 8a$ with respect to time. We get

$\frac{\mathrm{d}D}{\mathrm{dt}} = 4a\cos 4t + 4a\sin 4t.$

we let $\frac{\mathrm{d}D}{\mathrm{dt}} = 0, and\ we\ get$

$4a(\cos 4t + \sin 4t) = 0$ that is $1 + \tan 4t = 0$. This means $\tan 4t = -1$

So $4t = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots.$

Hence, $t = \frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \ldots.$

So, the first time that the 2 particles are furthest apart is when $t = \frac{3\pi}{16}$ and the maximum distance is

$D = a\sin\frac{12\pi}{16} - a\cos\frac{12\pi}{16} + 8a = \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}} + 8a = (\sqrt{2} + 8)a$

b) As we are rotating the shaded area $A_1$ about the $y$ axis, we must make $x$ the subject for the equations of the two log curves.

$y = \ln 2x \qquad\qquad y = \ln x$

$e^y = 2x \qquad\qquad e^y = x$

$x = \frac{1}{2}e^y$

The volume of solid $V_1$ is

$$V_1 = \pi\int_0^p (e^y)^2\,dy - \pi\int_0^p \left(\frac{1}{2}e^y\right)^2 dy$$

$$V_1 = \pi\int_0^p e^{2y}\,dy - \pi\int_0^p \frac{1}{4}\,e^{2y}\,dy$$

$$V_1 = \pi\left[\frac{1}{2}e^{2y}\right]_0^p - \pi\left[\frac{1}{8}e^{2y}\right]_0^p$$

$$V_1 = \pi\left[\frac{1}{2}e^{2p} - \frac{1}{2}\right] - \pi\left[\frac{1}{8}e^{2p} - \frac{1}{8}\right]$$

$$V_1 = \frac{\pi}{2}e^{2p} - \frac{\pi}{2} - \frac{\pi}{8}e^{2p} + \frac{\pi}{8}$$

$$V_1 = \left(\frac{3\pi}{8}e^{2p} - \frac{3\pi}{8}\right)\text{units}^3$$

The volume of solid $V_2$ is

$$V_2 = \pi\int_p^{\ln 3} (e^y)^2\,dy - \pi\int_p^{ln3}\left(\frac{1}{2}e^y\right)^2 dy$$

$$V_2 = \pi\int_p^{\ln 3} e^{2y}\,dy - \pi\int_p^{\ln 3}\frac{1}{4}\,e^{2y}\,dy$$

$$V_2 = \pi\left[\frac{1}{2}e^{2y}\right]_p^{\ln 3} - \pi\left[\frac{1}{8}e^{2y}\right]_p^{\ln 3}$$

$$V_2 = \pi\left[\frac{1}{2}e^{2\ln 3} - \frac{1}{2}\,e^{2p}\right] - \pi\left[\frac{1}{8}e^{2\ln 3} - \frac{1}{8}e^{2p}\right]$$

$$V_2 = \frac{\pi}{2}e^{2\ln 3} - \frac{\pi}{2}e^{2p} - \frac{\pi}{8}e^{2\ln 3} + \frac{\pi}{8}e^{2p}$$

$$V_2 = \frac{3\pi}{8}e^{2\ln 3} - \frac{3\pi}{8}e^{2p}$$

$$V_2 = \left(\frac{27\pi}{8} - \frac{3\pi}{8}e^{2p}\right)\text{units}^3$$

Now, as $V_2 = 3V_1$ then

$$\frac{27\pi}{8} - \frac{3\pi}{8}e^{2p} = 3\left(\frac{3\pi}{8}e^{2p} - \frac{3\pi}{8}\right)$$

$$\frac{27\pi}{8} - \frac{3\pi}{8}e^{2p} = \frac{9\pi}{8}e^{2p} - \frac{9\pi}{8}$$

Multiplying by 8, we get

$27\pi - 3\pi e^{2p} = 9\pi e^{2p} - 9\pi$

Dividing by $3\pi$, we get

$9 - e^{2p} = 3e^{2p} - 3$

$12 = 4e^{2p}$

$3 = e^{2p}$

$\ln 3 = \ln e^{2p}$

$2p = \ln 3$

Hence, $p = \frac{1}{2}\ln 3$.

c) i) As $x^{4n-1} + i = 0$ has roots $i, x_1, x_2, x_3, \ldots, x_{4n-2}$, then$(x-1)^{4n-1} + i = 0\ has\ roots\ 1 + i, 1 + x_1, \ldots$
$, 1 + x_{4n-3}, 1 + x_{4n-2}$.

OPTIONAL: <<To find an equation with roots $1 + i, 1 + x_1, 1 + x_2, 1 + x_3, \ldots, 1 + x_{4n-2}$, we let $y = 1 + x$ that is $x = y - 1$.

Now, by substituting $x = y - 1$ into the given equation we get $(y-1)^{4n-1} + i = 0$ and as the variable is immaterial, we can replace $y$ by $x$ and the equation will be $(x-1)^{4n-1} + i = 0$. >>

ii) Using a binomial expansion this equation can written as

$$\binom{4n-1}{0}x^{4n-1} + \binom{4n-1}{1}x^{4n-2}(-1) + \binom{4n-1}{2}x^{4n-2}(-1)^2$$
$$+ \cdots + \binom{4n-1}{4n-2}x(-1)^{4n-2} + \binom{4n-1}{4n-1}(-1)^{4n-1} + i = 0$$

Note: As $\binom{4n-1}{0} = 1, \binom{4n-1}{4n-1} = 1\ and\ (-1)^{4n-1} = -1$

then the equation can be written as

$$x^{4n-1} + \binom{4n-1}{1}x^{4n-2}(-1) + \binom{4n-1}{2}x^{4n-2}(-1)^2$$
$$+ \cdots + \binom{4n-1}{4n-2}x(-1)^{4n-2} - 1 + i = 0$$

Now, the roots of the equation above formed are $1 + i, 1 + x_1, 1 + x_2, 1 + x_3, \ldots, 1 + x_{4n-2}$. This means the product of these roots is

$$(1+i)(1+x_1)(1+x_2)(1+x_3)\ldots\ldots\ldots$$
$$\ldots\ldots\ldots.(1+x_{4n-2}) = -\frac{\text{constant term}}{\text{leading coefficient}}.$$

Note : As the power $4n - 1$ is odd then the number of terms in the binomial expansion is even so to find the product we divide the *negative* of constant term by the leading coefficient.

$So,\ (1+i)(1+x_1)(1+x_2)\ldots\ldots(1+x_{4n-2}) = \frac{-(-1+i)}{1}$

$(1+i)(1+x_1)(1+x_2)\ldots\ldots(1+x_{4n-2}) = 1 - i$

$(1+x_1)(1+x_2)(1+x_3)\ldots(1+x_{4n-2}) = \frac{1-i}{1+i}$

$(1+x_1)(1+x_2)(1+x_3)\ldots(1+x_{4n-2}) = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$

$(1+x_1)(1+x_2)(1+x_3)\ldots(1+x_{4n-2}) = -i$

d) Assume that $\log_7 5$ is rational. Let $\log_7 5 = \frac{p}{q}$, where $p$ and $q$ are positive integers with no common factor.

$\log_7 5 = \frac{p}{q}$ that is $5 = 7^{\frac{p}{q}}$ So, $5^q = 7^p$

Now since $q \geq 1$ it follows that 5 is a factor of the LHS. But clearly 5 is not a factor of the RHS as 5 is not factor of 7, nor a power of 7. Importantly, we have equality of two expressions whose prime factorisations are different. This contradicts the *Uniqueness Criterion* associated with the *Fundamental Theorem of Arithmetic*.

Hence $\log_7 5$ is not rational, that is, it must be irrational.

**Question 15**

a) i) $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1}$ is a finite geometric series with *n* terms where: $a = x^{n-1}$ *and* $r = -\frac{y}{x}$.

$$S_n = \frac{x^{n-1}\left(1-\left(\frac{-y}{x}\right)^n\right)}{1-\left(\frac{-y}{x}\right)}$$

$$S_n = \frac{x^{n-1}\left(1+\left(\frac{y}{x}\right)^n\right)}{1+\left(\frac{y}{x}\right)}$$

$$S_n = \frac{x^n + y^n}{x+y}$$

, after multiplying throughout by $x$.

ii) Let $x = 4$ *and* $y = 2$ in the above identity. Then $4^{n-1} - 4^{n-2} \times 2 + \cdots - 4 \times 2^{n-2} + \cdots + 2^{n-1} = \frac{4^n + 2^n}{4+2}$. That is: $4^n + 2^n = 6(4^{n-1} - 4^{n-2} \times 2 + \cdots + 2^{n-1})$ or equivalently $4^n + 2^n = 6M$ with $M = 4^{n-1} - 4^{n-2} \times 2 + \cdots + 2^{n-1}$; an integer

**Alternative method**

ii) *n* is a positive odd integer in $4^n + 2^n$ This indicates that *n* can be written as $n = 2p+1$ where *p* is a positive integer.

So, $4^n + 2^n$

$= 4^{2p+1} + 2^{2p+1}$

$= (4+2)(4^{2p} - 4^{2p-1} \times 2 + 4^{2p-2} \times 2^2 - \cdots + 2^{2p})$

$= 6(4^{2p} - 4^{2p-1} \times 2 + 4^{2p-2} \times 2^2 - \cdots + 2^{2p})$

Let $4^{2p} - 4^{2p-1} \times 2 + 4^{2p-2} \times 2^2 - \cdots + 2^{2p} = M$, where $M$ is a positive integer.

So, $4^n + 2^n = 6M$ which is divisible by 6.

Hence, if n is odd then $4^n + 2^n$ is divisible by 6.

**Alternative method, but INVALID approach – Teaching Purposes only**

Using mathematical induction

For $n = 1$, we get

$4^1 + 2^1 = 6$, which is divisible by 6.

Hence, the statement is true for $n = 1$

Assume the statement is true for $n = 2k+1$

that is $4^{2k+1} + 2^{2k+1} = 6P$, where $P$ is a positive integer.

For $n = 2k+3$

$4^{2(k+1)+1} + 2^{2(k+1)+1}$

$= 4^{2k+3} + 2^{2k+3}$

$= 4^2 \times 4^{2k+1} + 2^2 \times 2^{2k+1}$

$= 16 \times 4^{2k+1} + 4 \times 2^{2k+1}$

$= 12 \times 4^{2k+1} + 4 \times 4^{2k+1} + 4 \times 2^{2k+1}$

$= 12 \times 4^{2k+1} + 4(4^{2k+1} + 2^{2k+1})$

$= 12 \times 4^{2k+1} + 4 \times 6P$ (From assumption)

$= 6(2 \times 4^{2k+1} + 4P)$

Let $2 \times 4^{2k+1} + 4P = H$, where $H$ is a positive integer.

So, $4^{2k+3} + 2^{2k+3} = 6H$

Hence, if the statement is true for $n = 2k+1$ it is also true for $n = 2k+3$.

The statement was proved true for *n* = 1 and by mathematical induction it is true for *n* = 3, *n* = 5 and so on. Hence, it is true for all values of odd positive integers $n \geq 1$.

b) Consider the series $1 + \frac{1}{2}\alpha^3 + \frac{1}{4}\alpha^6 + \frac{1}{8}\alpha^9 + \cdots$

As $\frac{T_2}{T_1} = \frac{1}{2}\alpha^3$ and $\frac{T_3}{T_2} = \frac{\frac{1}{4}\alpha^6}{\frac{1}{2}\alpha^3} = \frac{1}{2}\alpha^3$

Hence, this series is a geometric series with infinite Terms; its first term is $a = 1$ and its common ratio

$$r = \frac{1}{2}\alpha^3 = \frac{1}{2}\left(\frac{1}{2}e^{i\theta}\right)^3 = \frac{1}{2} \times \frac{1}{8}e^{i3\theta} = \frac{1}{16}e^{i3\theta}$$

$So, r = \frac{1}{16}(\cos 3\theta + i \sin 3\theta)$.

Note : As $|r| = \frac{1}{16}$, that is, as $-1 < r < 1$, the limiting sum exists.

The sum to infinity is $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}\alpha^3}$

$$S_\infty = \frac{1}{1-\frac{1}{16}(\cos 3\theta + i \sin 3\theta)}$$

$$= \frac{16}{16 - (\cos 3\theta + i \sin 3\theta)}$$

$$= \frac{16}{16 - \cos 3\theta - i \sin 3\theta} \times \frac{16 - \cos 3\theta + i \sin 3\theta}{16 - \cos 3\theta + i \sin 3\theta}$$

$$= \frac{256 - 16\cos 3\theta + 16i \sin 3\theta}{256 - 32\cos 3\theta + \cos^2 3\theta + \sin^2 3\theta}$$

$$= \frac{256 - 16\cos 3\theta + 16i \sin 3\theta}{257 - 32\cos 3\theta}$$

Hence, the real part of the series is $\frac{256 - 16\cos 3\theta}{257 - 32\cos 3\theta}$.

c)

$\alpha\beta\gamma$ has a decimal expansion of $100\alpha + 10\beta + \delta$.

$As\ \alpha + \beta + \delta$ is divisible by 3, then

$\alpha + \beta + \delta = 3k$ where $k$ is an integer.

Now:$100\alpha + 10\beta + \delta = 99\alpha + 9\beta + \alpha + \beta + \delta$

$$= 3(33\alpha + 3\beta) + 3k$$

$$= 3(33\alpha + 3\beta + k)$$

which is divisible by 3.

d)

$$(a - b)^2 \geq 0$$

$$a^2 - 2ab + b^2 \geq 0$$

$$a^2 + b^2 \geq 2ab$$

$$(\div\ ab > 0)$$

$$\frac{a^2}{ab} + \frac{b^2}{ab} \geq 2$$

$$\frac{a}{b} + \frac{b}{a} \geq 2 \qquad (1)$$

Similarly,

$$a^2 + c^2 \geq 2ac \quad \rightarrow \quad \frac{a}{c} + \frac{c}{a} \geq 2 \qquad (2)$$

$$b + c^2 \geq 2bc \quad \rightarrow \quad \frac{b}{c} + \frac{c}{b} \geq 2 \qquad (3)$$

Adding (1), (2) and (3)

$$\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b} \geq 6$$

$$\frac{a+c}{b} + \frac{b+c}{a} + \frac{a+b}{c} \geq 6$$

e) i) We have from Newton's Second Law:

$$m\ddot{x} = -mg - mkv$$

$$\ddot{x} = -g - kv$$

$$\int_0^t \frac{dt}{dv} dv = -\int_{g/k}^{v} \frac{dv}{g + kv}$$

$$t = -\frac{1}{k}(ln(g + kv) - \ln(2g))$$

$$-kt = ln\left(\frac{g + kv}{2g}\right)$$

$$2ge^{-kt} = g + kv$$

$$g(2e^{-kt} - 1) = kv \quad \ldots (1)$$

ii) Again,

$$k\int_0^x \frac{dx}{dt} dt = \int_0^t g(2e^{-kt} - 1)dt$$

$$kx = g\left(-\frac{2}{k}e^{-kt} - t\right) + \frac{2g}{k}$$

$$k^2 x = g(2 - 2e^{-kt} - kt) \ldots (2)$$

Maximum height occurs when $v = 0$. This implies $e^{-kt} = \frac{1}{2}$ or equivalently, $kt = ln2$ and hence the result follows.

**Question 16**

a) For $n = 1$, LHS $= 3^1 = 3$ and RHS $= 1 + 3 - 1 = 3$

As LHS $\geq$ RHS then the statement is true for $n = 1$.

Assume the statement is true for $n = k$

that is $3^k \geq k^2 + 3k - 1$

Our aim is to prove the statement is true for $n = k + 1$

that is $3^{k+1} \geq (k + 1)^2 + 3(k + 1) - 1$

$3^{k+1} \geq k^2 + 2k + 1 + 3k + 3 - 1$

$3^{k+1} \geq k^2 + 5k + 3 \quad (1)$

Starting from the assumption and multiplying both sides by 3, we get $3^{k+1} \geq 3k^2 + 9k - 3$

Now, let us consider the function

$y = (3k^2 + 9k - 3) - (k^2 + 5k + 3)$

$y = 3k^2 + 9k - 3 - k^2 - 5k - 3$

$y = 2k^2 + 4k - 6$

$y = 2(k + 3)(k - 1)$ . This is a parabola which has a root $k = 1$ and is positive for any $k \geq 1$. This indicates that $y \geq 0$ for any $k \geq 1$, that is

$(3k^2 + 9k - 3) - (k^2 + 5k + 3) \geq 0$ for any $k \geq 1$

$3k^2 + 9k - 3 \geq k^2 + 5k + 3 \quad (2)$

From (1) and (2) we can deduce that

$3^{k+1} \geq 3k^2 + 9k - 3 \geq k^2 + 5k + 3$

Hence, if the statement is true for $n = k$ it is also true for $n = k + 1$.

The statement was proved true for $n = 1$ and by mathematical induction it is true for $n = 2$, $n = 3$ and so on.
Hence, it is true for all positive integers $n \geq 1$.

b) $tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$

As $\alpha$ and β are complementary angles, $tan\alpha + tan\beta > 0$ and $tan(\alpha+\beta)$ is undefined. Only valid conclusion is for $1 - tan\alpha tan\beta = 0$ and the result follows.

**Alternative method**

$\alpha + \beta = 90^{\circ}$

$\sin\theta = \cos(90^{\circ} - \theta)$
$\cos\theta = \sin(90^{\circ} - \theta)$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ .

$\tan\alpha \tan\beta = \tan\alpha \,.\, \tan(90^{\circ} - \alpha)$

$$= \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\sin(90^{\circ}-\alpha)}{\cos(90^{\circ}-\alpha)}$$

$$= \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\cos\alpha}{\sin\alpha}$$

$$= 1$$

c) i) Horizontal | Vertical

$x = V\,t\cos\alpha$ $\qquad$ $y = V\,t\sin\alpha - \frac{1}{2}\,gt^2$

Find the cartesian equation

$$t = \frac{x}{V\cos\alpha}$$

$$y = V\,\frac{x}{V\cos\alpha}\sin\alpha - \frac{1}{2}\,g\left(\frac{x}{V\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}$$

Since it passes through $(a, b)$

$$b = a\tan\alpha - \frac{g\,a^2\sec^2\alpha}{2V^2}$$

$$2bV^2 = 2aV^2\tan\alpha - g\,a^2\sec^2\alpha$$

$$2bV^2 = 2aV^2\tan\alpha - g\,a^2(1+\tan^2\alpha)$$

$$2bV^2 = 2aV^2\tan\alpha - g\,a^2 - g\,a^2\tan^2\alpha$$

$$g\,a^2\tan^2\alpha - 2aV^2\tan\alpha + 2bV^2 + g\,a^2 = 0$$

$$\tan^2\alpha - \frac{2aV^2\tan\alpha}{g\,a^2} + \frac{2bV^2 + g\,a^2}{g\,a^2} = 0$$

$$\tan^2\alpha - \frac{2V^2\tan\alpha}{g\,a} + \frac{2bV^2}{g\,a^2} + 1 = 0$$

ii) Let $m = \tan\alpha$

$m^2 - \frac{2V^2}{g\,a}m + \frac{2bV^2}{g\,a^2} + 1 = 0$ This is a quadratic in $m$ or equivalently, a reducible quadratic in tan α. Since α is acute, two trajectories will exist if the roots in the above quadratic equation are distinct and for distinct real roots:

$$b^2 - 4ac > 0$$

i.e.

$$\left(\frac{2V^2}{ga}\right)^2 - 4\left(\frac{2bV^2}{ga^2} + 1\right) > 0$$

$$(2V^2)^2 - 8bgV^2 - 4g^2a^2 > 0$$

$$8bgV^2 < (2V^2)^2 - 4g^2a^2$$

$$\text{that is, } b < \frac{V^2}{2g} - \frac{ga^2}{2V^2}.$$

Footnote: Thus, $(a, b)$ lies on, or inside the parabola

$$y = \frac{V^2}{2g} - \frac{gx^2}{2V^2}$$

There is one solution if it lies on the parabola and two distinct solutions if inside the parabola.

If there are two solutions for tan α then there are two solutions for α and hence two trajectories.

iii) If $b = 0$, the quadratic equation becomes:

$\tan^2\alpha - \frac{2V^2\tan\alpha}{g\,a} + 1 = 0$ . Let the roots be $a$ $and$ $b$ with corresponding angles $\alpha_1 and\ \alpha_2$. We note that the product of the roots is 1, that is, $ab = 1$ or equivalently: $tan\alpha_1 \times tan\alpha_2 = 1$. Appealing to the result in part (b) above completes the proof.

**Alternative method**

If the target is on the horizontal plane, $b = 0$, then the values of $\tan\alpha$ are the roots of

$$m^2 - \frac{2V^2}{ga}m + \frac{2(0)V^2}{ga^2} + 1 = 0 \quad \text{from } (i)$$

$$m^2 - \frac{2V^2}{ga}m + 1 = 0$$

Therefore:

$$m_1 = \frac{\frac{2V^2}{ga} + \sqrt{\left(\frac{2V^2}{ga}\right)^2 - 4}}{2} \quad \text{and} \quad m_2 = \frac{\frac{2V^2}{ga} - \sqrt{\left(\frac{2V^2}{ga}\right)^2 - 4}}{2}$$

$$m_1 m_2 = \frac{\left(\frac{2V^2}{ga}\right)^2 - \left[\left(\frac{2V^2}{ga}\right)^2 - 4\right]}{4} = \frac{4}{4} = 1$$

From (b) above and since $m_1 m_2 = 1$ then the values of $\alpha$ are complementary.

d)

$$abcd + efgh = abcd - cdgh + cdgh + efgh - abef + abef$$

$$= cd\,( ab - gh) + cdgh - ef( ab - gh) + abef$$

$$= ( ab - gh)( cd - ef) + abef + cdgh \quad (1)$$

Now, as $ab - gh$ is a divisor of $abef + cdgh$ then

$abef + cdgh = M( ab - gh)$ (2), where $M$ is a pronumeral; though potentially an integer.
By substituting (2) in (1), we get
$abcd + efgh$

$$= ( ab - gh)( cd - ef) + M( ab - gh)$$

$$= ( ab - gh)( cd - ef + M)$$

Hence, $ab - gh$ is a divisor of $abcd + efgh$.

Multiple Choice Summary Answers

| | |
|---|---|
| **1. C** | **6. D** |
| **2. B** | **7. C** |
| **3. A** | **8. A** |
| **4. D** | **9. A** |
| **5. B** | **10. B** |

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